Two definitions of connections

(M, ∇) $\quad \nabla: \mathfrak{X}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$

$\qquad\qquad\qquad (X, K) \longmapsto \nabla_X K$

- ∇_X - $\mathbb{R}$ linear in K and preserves type
- $\nabla_{fX+gY} K = f \nabla_X K + g \nabla_Y K$
- $\nabla_X (K \otimes L) = \nabla_X K \otimes L + K \otimes \nabla_X L$
- commutes with contractions
- $\nabla_X (f) = X(f)$.

Second definition in terms of charts.

In particular

$$\nabla_{X_\mu} X_\nu = \Gamma^\rho{}_{\nu\mu}(\omega) X_\rho$$

connection coefficients

Connections and parallelism :

Proposition

$$\left(\begin{array}{l} M\text{-manifold with } \nabla \\ \gamma : I \longrightarrow M \text{ differentiable curve} \\ V\text{- vector field along } \gamma \end{array}\right) \Longrightarrow$$

$\exists!\ \frac{DV}{dt}$ another vector field along γ with the following properties:

$$1^\circ \quad \frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$$

$$2^\circ \quad \frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$$

3° If V is induced by a vector field $\tilde{V}$ on M then

$$\frac{DV}{dt} = \nabla_{\frac{d\gamma}{dt}} \tilde{V}$$

Proof

First let us assume that such $\frac{D}{dt}$ exists.

$$(\mathcal{U}, x), \quad X_\mu = \frac{\partial}{\partial x^\mu} \Longrightarrow V = V^\mu X_\mu$$

$$\frac{DV}{dt} = \frac{D}{dt}(V^\mu X_\mu) = \frac{dV^\mu}{dt} X_\mu + V^\mu \frac{DX_\mu}{dt} =$$

$$= \frac{dV^\mu}{dt} X_\mu + V^\mu \nabla_{\frac{d\gamma}{dt}} X_\mu = \frac{dV^\mu}{dt} + V^\mu \frac{d\cancel{\gamma}^\alpha}{dt} \nabla_{X_\alpha} X_\mu =$$

$$= \frac{dV^\mu}{dt} X_\mu + V^\mu \frac{d\cancel{\gamma}^\alpha}{dt} \Gamma^\beta_{\mu\alpha} X_\beta =$$

$$= \underbrace{\left(\frac{dV^\beta}{dt} + \Gamma^\beta_{\mu\alpha} V^\mu \frac{d\cancel{x}^\alpha}{dt} \right)}_{} X_\beta \qquad (*)$$

this are local components of $\frac{DV}{dt}$.

$$\boxed{\text{So if } \frac{DV}{dt} \text{ exists it is unique}}$$

Now locally we define $\frac{DV}{dt}$ by $(*)$.

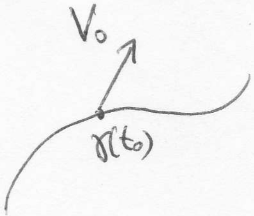
If we take another $(\mathcal{U}', x')$ and define $\frac{DV}{dt}$ by $(*)$. The def. agree on the overlap by uniqueness.

Definition

(M, ∇). Vector field V along $\gamma : I \to M$ is called parallel if $\frac{DV}{dt} = 0$, $\forall t \in I$.

Proposition

(M, ∇). $\gamma : I \to M$ and let V_0 be a vector tangent to M at $\gamma(t_0)$, $t_0 \in I$.



Then there exists a unique vector field V which is parallel along γ ~~and~~ such that

$$V(t_0) = V_0.$$

Proof

In one coordinate chart ~~along~~ around $C(t_0)$ $V(t)$ satisfies

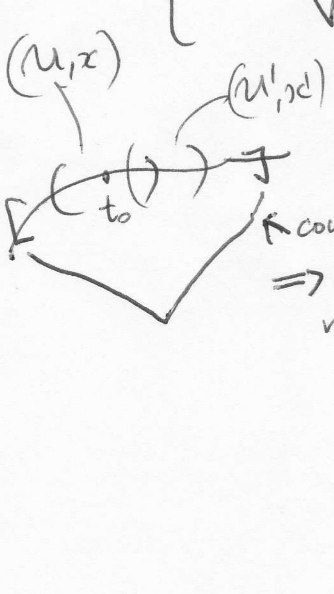
$$0 = \frac{DV}{dt} = \left[\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} V^\nu \frac{dx^\rho}{dt}\right] X_\mu$$

Thus coordinates of V should satisfy

$$\begin{cases} \dfrac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} V^\nu \dfrac{dx^\rho}{dt} = 0 \\ V^\mu(t_0) = V^\mu_0 \end{cases}$$

linear differential equation for V^μ with initial cond $V^\mu(t_0) = V^\mu_0$.

has a unique solution for all t



compact

$\Rightarrow$ finite number of (U, x) is enough. $\square$

Second definition

$$\nabla \rightsquigarrow \Gamma^\mu{}_{\nu\varrho} \rightsquigarrow \Gamma^\mu{}_\nu(\omega)$$

$\mathcal{U}$ and $\mathcal{U}'$ two open sets s.t. $\mathcal{U}\cap\mathcal{U}'\neq\emptyset$.

ω^μ coframe in $\mathcal{U}$

ω'^μ coframe in $\mathcal{U}'$

We define connection in $\mathcal{U}$ by connection 1-forms $\Gamma^\mu{}_\nu$ in $\mathcal{U}$. Connection 1-forms $\Gamma'^\mu{}_\nu(\omega)$ ~~define~~ in $\mathcal{U}'$ define the same connection in $\mathcal{U}\cup\mathcal{U}'$ iff there exists a function $a: \mathcal{U}\cap\mathcal{U}' \to GL(n,\mathbb{R})$ s.t.

$$\Gamma'^\mu{}_\nu(\omega) = a^\mu{}_\alpha \Gamma^\alpha{}_\beta(\omega) a^{-1\beta}{}_\nu + da^\mu{}_\alpha a^{-1\alpha}{}_\nu$$

<u>k-form of type ϱ</u>: $\varrho: GL(n,\mathbb{R}) \xrightarrow{\text{hom.}} GL(N,\mathbb{R})$, $W=\mathbb{R}^N$.

$$\overset{k}{\alpha}: \omega \longmapsto \alpha^k(\omega) \in W\otimes\Lambda^k M$$

$$\overset{k}{\alpha}(a\omega) = \varrho(a)\overset{k}{\alpha}(\omega).$$

$$d\overset{k}{\alpha}(a\omega) \neq \varrho(a)\, d\overset{k}{\alpha}(\omega)$$ (except for scalar forms $\varrho(a)=1$, $N=1$)

D extension of d s.t.

$$D\overset{k}{\alpha}(a\omega) = \varrho(a)\, d\overset{k}{\alpha}(\omega).$$

ex

$$DX^\mu = dX^\mu + \Gamma^\mu{}_\nu \wedge X^\nu$$ X^μ - k-form of type id // $\varrho(a)=a$

$$D\Omega^\mu{}_\nu = d\Omega^\mu{}_\nu + \Gamma^\mu{}_\alpha\wedge\Omega^\alpha{}_\nu - \Gamma^\alpha{}_\nu\wedge\Omega^\mu{}_\alpha$$ $\Omega^\mu{}_\nu$ k-form of type Ad

$$[Ad(a)\Omega]^\mu{}_\nu = a^\mu{}_\alpha \Omega^\alpha{}_\beta a^{-1\beta}{}_\nu$$

⋮

e.t.c.

Check: $\Omega^\mu{}_\nu(a\omega) = a^\mu{}_\alpha \Omega^\alpha{}_\beta(\omega) a^{-1\beta}{}_\nu$

$\Rightarrow D\Omega^\mu{}_\nu(a\omega) = a^\mu{}_\alpha D\Omega^\alpha{}_\beta(\omega) a^{-1\beta}{}_\nu$, because $\Gamma'(a\omega) = a\Gamma(\omega)a^{-1} + da\,a^{-1}$. ~~$\Gamma'(a\omega) = a\Gamma a^{-1} + da^{-1}$~~

$$\rho : GL(n,\mathbb{R}) \xrightarrow{\text{homo}} GL(N,\mathbb{R})$$

$$\rho' : \operatorname{End}\mathbb{R}^n \xrightarrow{\text{homo}} \operatorname{End}\mathbb{R}^N$$

$$A \in \operatorname{End}\mathbb{R}^n \Rightarrow a = \exp(tA) \in GL(n,\mathbb{R})$$

$$\rho'(A) = \frac{d}{dt}\,\rho(\exp(tA))\Big|_{t=0}. \in \operatorname{End}(\mathbb{R}^N)$$

$\mathbb{R}^n \ni v^\mu \quad \mu = 1, \dots n$ $\qquad a = (a^\mu{}_\nu) \in GL(n,\mathbb{R})$

$\mathbb{R}^N \ni \mathbf{v}^A \quad A = 1, \dots N$ $\qquad \rho(a)^A{}_B \in GL(N,\mathbb{R})$

$$\frac{d}{dt}\,\rho(\exp(tA))^A{}_B\Big|_{t=0} = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1} A^\mu{}_\nu$$

$$= \rho^A{}_B{}^\nu{}_\mu\, A^\mu{}_\nu \;=:\; \rho'(A)^A{}_B$$

$$\boxed{\rho'(A)^A{}_B = \rho^A{}_B{}^\nu{}_\mu\, A^\mu{}_\nu \,, \quad \rho^A{}_B{}^\nu{}_\mu = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1}}$$

α – k-form of type ρ

$\Rightarrow$

$$D\alpha = d\alpha + \rho'(\Gamma) \wedge \alpha$$

$$\boxed{(D\alpha)^A(\omega) = d\,\alpha^A(\omega) + \rho^A{}_B{}^\nu{}_\mu\, \Gamma^\mu{}_\nu(\omega) \wedge \alpha^B(\omega)}$$